

Frames and their relatives and reproducing kernel Hilbert spaces

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Abstract

This paper is devoted to three aspects of the relation between reproducing kernel Hilbert spaces (RKHS) and stable analysis/synthesis processes: First, we analyze the structure of the reproducing kernel of a RKHS using frames and reproducing pairs. Second, we give a new proof for the result that finite redundancy of a continuous frame implies atomic structure of the underlying measure. Our proof relies on the RKHS structure of the range of the analysis operator. This result also implies that every continuous Riesz basis can in fact be identified with a discrete Riesz basis. Finally, we show how the range of the analysis operators of a reproducing pair can be equipped with a RKHS structure.

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1 Introduction

Reproducing kernel Hilbert spaces were introduced by Zarembo [39] and Mercer [29] and were first studied in a systematic fashion by Aronszajn [10] in 1950. Ever since these spaces have played an important role in many branches of mathematics such as complex analysis [20], approximation theory [38] and, only recently, in learning theory and classification due to the celebrated representer theorem [33]. Another field with manifold connections to reproducing kernels is frame theory and its relatives.

Discrete frames have been introduced in the 1950's in the context of nonharmonic Fourier analysis [19] and have then been generalized to continuous frames on arbitrary positive measure spaces in the early 1990's [2, 28].

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Reproducing kernel theory can be employed to construct continuous frames and conversely frame theory can be used to study reproducing kernels [27].

Although frames are convenient objects to work with, there exists a large reservoir of interesting systems that are complete and do not satisfy both frame conditions. Therefore, semi-frames [4, 6] and reproducing pairs [7, 36, 37] have been introduced. An upper (resp. lower) semi-frame is a complete system that only satisfies the upper (resp. lower) frame inequality. A reproducing pair is a pair of mappings that generates a bounded and boundedly invertible analysis/synthesis process without assuming any frame inequality.

This paper is divided into three major parts portraying different connections between frames, reproducing pairs and reproducing kernel Hilbert spaces. In the first part we investigate systems taking values in a reproducing kernel Hilbert space. We present an explicit expression for the reproducing kernel in terms of a reproducing pair. This is an extension of the results from [30, 32]. Moreover, we introduce a novel necessary condition for a vector family to form a frame.

The second part is devoted to studying the redundancy of (semi-)frames. In the discrete case, the redundancy of a frame measures how much the Hilbert space is oversampled by the frame, see for example [15, 16]. It is however impossible to directly generalize the notion of redundancy to continuous (semi-) frames. The approach chosen in [11] thus takes a detour via the concept of Riesz bases, i.e., non-redundant discrete frames. A discrete frame Ψ is a Riesz basis if its analysis operator C_Ψ is surjective. Following [11], the redundancy of a (semi-)frame is defined by

$$R(\Psi) := \dim(\text{Ran } C_\Psi^\perp).$$

It has been observed in several articles [11, 25, 26] that $R(\Psi)$ depends on the underlying measure space (X, μ) . In particular, if a (lower semi-)frame has finite redundancy, then it follows that (X, μ) is atomic. The proofs in the aforementioned papers all rely in one way or the other on the following argument: If the redundancy of a frame is zero (finite), then

$$\inf \{ \mu(A) : A \text{ measurable and } \mu(A) > 0 \} = C > 0,$$

which implies that (X, μ) is atomic. We will give a new proof here using the reproducing kernel Hilbert space structure of the range of C_Ψ . It is interesting to note that upper semi-frames behave essentially different in this regard. We show that there exist upper semi-frames on non-atomic measure spaces with redundancy zero.

As a side product, we conclude that efforts to generalize Riesz bases to the continuous setting [9, 22] cannot succeed. This is because the underlying measure space of a frame with redundancy zero is atomic (and therefore

discrete). Moreover, we show that every frame can be split into a discrete and a strictly continuous Bessel system.

The final part of this paper is concerned with characterizing the ranges of the analysis operators of a reproducing pair. The omission of the frame inequalities causes the problem that $\text{Ran } C_\Psi$ need no longer be contained in $L^2(X, \mu)$. We will demonstrate how a reproducing pair intrinsically generates a pair of reproducing kernel Hilbert spaces and calculate the reproducing kernel.

This paper is organized as follows. After introducing the main concepts in Section 2 we first consider systems on reproducing kernel Hilbert spaces in Section 3. Then, in Section 4 we investigate the redundancy of continuous (semi-)frames. Finally, we show how a reproducing pair intrinsically generates a pair of RKHSs in Section 5 and characterize the reproducing kernels.

2 Preliminaries

2.1 Atomic and non-atomic measures

Throughout this paper we will assume that (X, μ) is a nontrivial measure space with μ being σ -finite and positive. A measurable set $A \subset X$ is called an atom if $\mu(A) > 0$ and for any measurable subset $B \subset A$, with $\mu(B) < \mu(A)$, it holds $\mu(B) = 0$. A measure space is called atomic if there exists a partition $\{A_n\}_{n \in \mathbb{N}}$ of X consisting of atoms and null sets. (X, μ) is called non-atomic if there are no atoms in (X, μ) . To our knowledge there is no term to denote a measure space which is not atomic. In order to avoid any confusion with non-atomic spaces, we will therefore call a measure space an-atomic if it is not atomic.

A well-known result by Sierpiński states that non-atomic measures take a continuity of values.

Theorem 1 (Sierpiński [34]) *Let (X, μ) be a non-atomic measure space and let $A \subset X$ be measurable with positive measure, then, for every $0 \leq b \leq \mu(A)$, there exists $B \subset A$ such that $\mu(B) = b$.*

We will later separate the purely continuous part of a frame from the discrete part. For the construction, we need the following auxiliary result. Since we could not find any reference for the second part, we will provide a proof in the appendix.

Lemma 2 *Let (X, μ) be a σ -finite measure space.*

(i) *There exists μ_a atomic and μ_c non-atomic such that*

$$\mu = \mu_a + \mu_c. \tag{1}$$

(ii) *If (X, μ) is an-atomic, then there exists $A \subset X$ with $\mu(A) > 0$ and (A, μ) non-atomic.*

2.2 Continuous frames, semi-frames and reproducing pairs

Frames were first introduced by Duffin and Schaeffer [19] in the context of non-harmonic Fourier analysis. In the early 1990's, Ali et al. [2] and Kaiser [28] independently extended frames to mappings acting on a measure space (X, μ) .

Denote by $GL(\mathcal{H})$ the space of all bounded linear operators on \mathcal{H} with bounded inverse and let \mathcal{H} be a separable Hilbert space.

Definition 1 *A mapping $\Psi : X \rightarrow \mathcal{H}$ is called a continuous frame if*

(i) *Ψ is weakly measurable, that is, $x \mapsto \langle f, \Psi(x) \rangle$ is a measurable function for every $f \in \mathcal{H}$,*

(ii) *there exist positive constants $m, M > 0$ such that*

$$m \|f\|^2 \leq \int_X |\langle f, \Psi(x) \rangle|^2 d\mu(x) \leq M \|f\|^2, \quad \forall f \in \mathcal{H}. \quad (2)$$

The constants m, M are called the frame bounds and Ψ is called Bessel if at least the second inequality in (2) is satisfied. If (X, μ) is a countable set equipped with the counting measure then one recovers the classical definition of a discrete frame, see for example [17]. For a short and self-contained introduction to continuous frames, we refer the reader to [31].

The fundamental operators in frame theory are the analysis operator $C_\Psi : \mathcal{H} \rightarrow L^2(X, \mu)$, $C_\Psi f(x) := \langle f, \Psi(x) \rangle$, and the synthesis operator

$$D_\Psi : L^2(X, \mu) \rightarrow \mathcal{H}, \quad D_\Psi F := \int_X F(x) \Psi(x) d\mu(x),$$

where the integral is defined weakly. Observe that $C_\Psi^* = D_\Psi$ whenever Ψ is Bessel. The frame operator $S_\Psi \in GL(\mathcal{H})$ is defined as the composition of C_Ψ and D_Ψ

$$S_\Psi : \mathcal{H} \rightarrow \mathcal{H}, \quad S_\Psi f := D_\Psi C_\Psi f = \int_X \langle f, \Psi(x) \rangle \Psi(x) d\mu(x).$$

Every frame Ψ^d satisfying

$$f := D_\Psi C_{\Psi^d} f = D_{\Psi^d} C_\Psi f, \quad \forall f \in \mathcal{H},$$

is called a dual frame for Ψ . For every frame there exists at least one dual frame $S_\Psi^{-1} \Psi$, called the canonical dual frame. As the analysis operator is in general not onto $L^2(X, \mu)$, there may exist several dual frames for Ψ .

Frames have proven to be a useful tool in many different fields of mathematics such as signal processing [14] or mathematical physics [3, 13]. There is however a great variety of examples of complete systems that do not meet both frame conditions. Several concepts to generalize the frame property

have thus been proposed. An upper (resp. lower) semi-frame is a complete system that only satisfies the upper (resp. lower) frame inequality, see [4, 5, 6].

Another generalization is the concept of reproducing pairs, defined in [36] and further investigated in [7, 8, 37]. Here, one considers a pair of mappings instead of a single one and no frame inequality is assumed to hold.

Definition 2 *Let $\Psi, \Phi : X \rightarrow \mathcal{H}$ weakly measurable. The pair of mappings (Ψ, Φ) is called a reproducing pair for \mathcal{H} if the resolution operator $S_{\Psi, \Phi} : \mathcal{H} \rightarrow \mathcal{H}$, weakly defined by*

$$\langle S_{\Psi, \Phi} f, g \rangle := \int_X \langle f, \Psi(x) \rangle \langle \Phi(x), g \rangle d\mu(x), \quad (3)$$

is an element of $GL(\mathcal{H})$.

Observe that Definition 2 is indeed a generalization of continuous frames. On the one hand, neither Ψ nor Φ are required to meet the frame conditions and, on the other hand, a weakly measurable mapping Ψ is a continuous frame if, and only if, (Ψ, Ψ) is a reproducing pair. Note that reproducing pairs are conceptually similar to the concept of weak duality [21] where one considers expansions in terms of a Gelfand triplet.

2.3 Reproducing kernel Hilbert spaces (RKHS)

Let $\mathcal{F}(X, \mathbb{C})$ denote the vector space of all functions $f : X \rightarrow \mathbb{C}$. Reproducing kernel Hilbert spaces are in a way convenient subspaces of $\mathcal{F}(X, \mathbb{C})$ since they allow for pointwise interpretation of functions, unlike for example Lebesgue spaces.

Definition 3 *Let $\mathcal{H}_K \subset \mathcal{F}(X, \mathbb{C})$ be a Hilbert space, \mathcal{H}_K is called a reproducing kernel Hilbert space (RKHS) if the point evaluation functional $\delta_x : \mathcal{H}_K \rightarrow \mathbb{C}$, $\delta_x(f) := f(x)$ is bounded for every $x \in X$, that is, if there exists $C_x > 0$ such that $|\delta_x(f)| \leq C_x \|f\|$, for all $f \in \mathcal{H}_K$.*

As δ_x is bounded, there exists a unique vector $k_x \in \mathcal{H}_K$ such that $f(x) = \langle f, k_x \rangle$, for all $f \in \mathcal{H}_K$. The function $K(x, y) = k_y(x) = \langle k_y, k_x \rangle$ is called the reproducing kernel for \mathcal{H}_K . The reproducing kernel is unique, $K(x, y) = \overline{K(y, x)}$ and its diagonal is of the following form

$$K(x, x) = \langle k_x, k_x \rangle = \|k_x\|^2 = \sup \{ |f(x)|^2 : f \in \mathcal{H}_K, \|f\| = 1 \}.$$

The following result can be found in [2, Theorem 3.1].

Theorem 3 *Let \mathcal{H}_K be a RKHS and $\{\phi_i\}_{i \in \mathcal{I}} \subset \mathcal{H}_K$ an orthonormal basis, then*

$$K(x, y) = \sum_{i \in \mathcal{I}} \phi_i(x) \overline{\phi_i(y)}, \quad (4)$$

with pointwise convergence. In particular,

$$0 < \sum_{i \in \mathcal{I}} |\phi_i(x)|^2 = K(x, x) < \infty, \quad \forall x \in X. \quad (5)$$

Conversely, if there exists an orthonormal basis for a Hilbert space $\mathcal{H}_K \subset \mathcal{F}(X, \mathbb{C})$ that satisfies (5), then \mathcal{H}_K can be identified with a RKHS consisting of functions $f : X \rightarrow \mathbb{C}$.

If X is equipped with a measure μ and $\mathcal{H}_K \subset L^2(X, \mu)$, then $\Psi(x) := K(x, \cdot)$ is a continuous Parseval frame.

For a thorough introduction to RKHS we refer the reader to [10, 30]. We will investigate the connection between RKHS and frames (resp. reproducing pairs) in two different ways. In Section 3 we consider frames (resp. reproducing pairs) taking values in a RKHS, whereas in Section 4 and 5 we investigate the RKHS generated by the range of the analysis operator of a frame (resp. reproducing pair).

3 Frames and reproducing pairs taking values in a RKHS

In this section we will mainly investigate two questions. First, given a RKHS, what can be said about the pointwise behavior of frames and how can the reproducing kernel be characterized. Second, which conditions on a frame ensure that the space possesses a reproducing kernel.

The following result adapts the arguments of the proof of [30, Theorem 3.12] to the case of reproducing pairs.

Theorem 4 *Let \mathcal{H}_K be a RKHS and $\Psi = \{\phi_i\}_{i \in \mathcal{I}}$, $\Phi = \{\psi_i\}_{i \in \mathcal{I}} \subset \mathcal{H}_K$. The pair (Ψ, Φ) is a reproducing pair for \mathcal{H}_K if, and only if, there exists $A \in GL(\mathcal{H}_K)$ such that*

$$K(x, y) = \sum_{i \in \mathcal{I}} (A\phi_i)(x) \overline{\psi_i(y)} = \sum_{i \in \mathcal{I}} (A^*\psi_i)(x) \overline{\phi_i(y)}, \quad (6)$$

where the series converges pointwise. In particular, A is unique and given by $S_{\Psi, \Phi}^{-1}$.

Proof: Let (Ψ, Φ) be a reproducing pair, it holds

$$K(x, y) = \langle k_y, k_x \rangle = \sum_{i \in \mathcal{I}} \langle k_y, \psi_i \rangle \langle S_{\Psi, \Phi}^{-1} \phi_i, k_x \rangle = \sum_{i \in \mathcal{I}} \overline{\psi_i(y)} (S_{\Psi, \Phi}^{-1} \phi_i)(x).$$

Conversely, assume that K is given by (6). Let $f, g \in \text{span}\{k_x : x \in X\}$, that is, there exist $\alpha_n, \beta_m \in \mathbb{C}$ such that $f = \sum_n \alpha_n k_{x_n}$ and $g = \sum_m \beta_m k_{y_m}$, then

$$\langle f, g \rangle = \sum_{n, m=1}^N \alpha_n \overline{\beta_m} \langle k_{x_n}, k_{y_m} \rangle = \sum_{n, m=1}^N \alpha_n \overline{\beta_m} K(y_m, x_n)$$

$$\begin{aligned}
&= \sum_{n,m=1}^N \alpha_n \overline{\beta_m} \sum_{i \in \mathcal{I}} (A\phi_i)(y_m) \overline{\psi_i(x_n)} = \sum_{n,m=1}^N \alpha_n \overline{\beta_m} \sum_{i \in \mathcal{I}} \langle k_{x_n}, \psi_i \rangle \langle A\phi_i, k_{y_m} \rangle \\
&= \sum_{i \in \mathcal{I}} \left\langle \sum_{n=1}^N \alpha_n k_{x_n}, \psi_i \right\rangle \left\langle A\phi_i, \sum_{m=1}^N \beta_m k_{y_m} \right\rangle \\
&= \sum_{i \in \mathcal{I}} \langle f, \psi_i \rangle \langle A\phi_i, g \rangle = \langle AS_{\Psi, \Phi} f, g \rangle.
\end{aligned}$$

In [30, Proposition 3.1] it is shown that $\text{span}\{k_x : x \in X\}$ is dense in \mathcal{H}_K . Therefore, it follows that $AS_{\Psi, \Phi} = I$. As $A \in GL(\mathcal{H}_K)$ we may conclude that $S_{\Psi, \Phi} \in GL(\mathcal{H}_K)$, that is, (Ψ, Φ) is a reproducing pair. \square

Remark 5 Results have been proven if Ψ and Φ are dual frames [32, Theorem 7] or if $\Psi = \Phi$ is a Parseval frame [30, Theorem 3.12], which are just particular cases of Theorem 4. In both cases one has $A = I$.

Proposition 6 Let \mathcal{H}_K be a RKHS and $\{\psi_i\}_{i \in \mathcal{I}} \subset \mathcal{H}_K$ Bessel, then it holds

$$\sum_{i \in \mathcal{I}} |\psi_i(x)|^2 < \infty, \quad \forall x \in X. \quad (7)$$

If $\{\psi_i\}_{i \in \mathcal{I}} \subset \mathcal{H}_K$ is a frame, then

$$0 < \sum_{i \in \mathcal{I}} |\psi_i(x)|^2 < \infty, \quad \forall x \in X. \quad (8)$$

Proof: Let $\{\psi_i\}_{i \in \mathcal{I}}$ be Bessel, then, for every $x \in X$, it holds

$$\sum_{i \in \mathcal{I}} |\psi_i(x)|^2 = \sum_{i \in \mathcal{I}} |\langle k_x, \psi_i \rangle|^2 \leq M \|k_x\|^2 < \infty.$$

An analogue argument shows the lower bound in (8) if $\{\psi_i\}_{i \in \mathcal{I}}$ is a frame. \square

Remark 7 (i) Observe that (8) is not a direct consequence of Theorem 4 as (6) only ensures

$$0 < K(x, x) = \sum_{i \in \mathcal{I}} (S_{\psi}^{-1} \psi_i)(x) \overline{\psi_i(x)} < \infty.$$

(ii) The converse of Proposition 6 is not true. To see this, consider $\ell^2(\mathbb{N})$. Every closed subspace of $\ell^2(\mathbb{N})$ is a RKHS. Define

$$\{\Delta_n\}_{n \in \mathbb{N}} := \{\delta_1, 2\delta_2, 1/3\delta_3, 4\delta_4, 1/5\delta_5, \dots\},$$

where $\delta_n[k] = \delta_{n,k}$ denotes the canonical basis in $\ell^2(\mathbb{N})$. Then

$$\sum_{n \in \mathbb{N}} |\Delta_n[k]|^2 = \begin{cases} k, & \text{if } k \text{ even} \\ 1/k, & \text{if } k \text{ odd.} \end{cases},$$

and (8) is satisfied. However, $\{\Delta_n\}_{n \in \mathbb{N}}$ satisfies neither the upper nor the lower frame inequality.

(iii) Another consequence of Proposition 6 is that, if $\{\psi_i\}_{i \in \mathcal{I}} \subset \mathcal{H}_K$ is a Bessel sequence, then

$$\left\| \sum_{i \in \mathcal{I}} \psi_i(x) \psi_i \right\| < \infty, \quad \forall x \in X.$$

Let us now consider the converse part of Theorem 3, that is starting from an orthonormal basis and give a condition that ensures that the space is a RKHS. We generalize the result to a condition on discrete frames.

Theorem 8 *Let $\mathcal{H}_K \subset \mathcal{F}(X, \mathbb{C})$ be a Hilbert space. If there exists a discrete frame Ψ for \mathcal{H}_K such that*

$$0 < \sum_{i \in \mathcal{I}} |\psi_i(x)|^2 = C_x < \infty, \quad \forall x \in X, \quad (9)$$

then \mathcal{H}_K is a RKHS in $\mathcal{F}(X, \mathbb{C})$.

Proof: Let Ψ be a frame such that (9) holds, then

$$\sum_{i \in \mathcal{I}} \overline{\psi_i(x)} S_{\Psi}^{-1} \psi_i \quad (10)$$

is well-defined in \mathcal{H}_K for every $x \in X$. Moreover, let $f \in \mathcal{H}_K$, then

$$\begin{aligned} |f(x)| &= \left| \sum_{i \in \mathcal{I}} \langle f, S_{\Psi}^{-1} \psi_i \rangle \psi_i(x) \right| = \left| \left\langle f, \sum_{i \in \mathcal{I}} \overline{\psi_i(x)} S_{\Psi}^{-1} \psi_i \right\rangle \right| \\ &\leq \|f\| \left\| \sum_{i \in \mathcal{I}} \overline{\psi_i(x)} S_{\Psi}^{-1} \psi_i \right\| = \sqrt{C_x M} \|f\|. \end{aligned}$$

Hence, point evaluation is continuous. \square

Observe that, all results in this section can be reformulated for continuous frames and reproducing pairs mapping from a measure space (Y, μ) to \mathcal{H}_K .

4 Continuous (semi-)frames and their redundancy

The redundancy of a discrete frame measures, roughly speaking, how much the frame is oversampling the Hilbert space \mathcal{H} . In order to introduce a notion of redundancy for continuous frames one may look at the properties that characterize non-redundant discrete frames. A non-redundant discrete frame is in fact a Riesz basis, that is, a frame with the additional property that $\text{Ran } C_\Psi = (\text{Ker } D_\Psi)^\perp = \ell^2(\mathcal{I})$. This motivates to define the redundancy $R(\Psi)$ by

$$R(\Psi) := \dim(\text{Ran } C_\Psi^\perp). \quad (11)$$

Note that C_Ψ is a Fredholm operator if Ψ is Bessel and has finite redundancy [1]. In that case, $R(\Psi) = -\text{ind}(C_\Psi) = \text{ind}(D_\Psi)$, where $\text{ind}(A)$ denotes the Fredholm index of a bounded operator A .

Let us briefly discuss and calculate this notion of redundancy for discrete and finite examples. Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis and define $\Psi = \{e_1\} \cup \{e_n\}_{n \in \mathbb{N}}$ and $\Phi = \{e_n\}_{n \in \mathbb{N}} \cup \{e_n\}_{n \in \mathbb{N}}$. Then $R(\Psi) = 1$ and $R(\Phi) = \infty$. In general one has for two discrete systems that $\Psi \subseteq \Phi$ implies $R(\Psi) \leq R(\Phi)$. If $\Psi = \{f_n\}_{n=1}^N$ is a finite frame for \mathbb{C}^d , then $R(\Psi) = N - d$, whereas the classical definition of redundancy for finite frames amounts to N/d .

4.1 Frames and lower semi-frames

The main goal of this section is to give a new proof of a result on the redundancy of continuous frames and its connection to the underlying measure space which has been stated in [25, Theorem 2], [11, Theorem 2.2] and [26, Proposition 3.3]. We thereby use the property that $\text{Ran } C_\Psi$ forms a RKHS. In our opinion, this proof better explains the inherent structure of continuous frames.

Theorem 9 *Let (X, μ) be a measure space and Ψ a (lower semi-)frame with nontrivial $\text{Dom } C_\Psi$. If the mapping Ψ has finite redundancy $R(\Psi) < \infty$, then (X, μ) is atomic.*

The converse is obviously not true, take for example the discrete family Φ from above.

Corollary 10 *Let (X, μ) be an-atomic and Ψ a frame, then $R(\Psi) = \infty$ and there exist infinitely many dual frames for Ψ .*

We need some preliminary results for the proof of Theorem 9.

Proposition 11 ([22], Corollary 2.9) *Let \mathcal{H}_K be a subspace of $L^2(X, \mu)$. If Ψ satisfies the lower frame inequality, then $(\text{Ran } C_\Psi, \|\cdot\|_2)$ is a RKHS. Moreover, the following are equivalent:*

- (i) *There exists a continuous frame Ψ , such that $\text{Ran } (C_\Psi) = \mathcal{H}_K$.*

(ii) \mathcal{H}_K is a RKHS.

Let Ψ be a frame, the reproducing kernel for $\text{Ran } C_\Psi$ is given explicitly by $K_\Psi(x, y) := \langle S_\Psi^{-1}\Psi(y), \Psi(x) \rangle$. Consequently, for $F \in L^2(X, \mu)$ there exists $f \in \mathcal{H}$, such that $F = C_\Psi f$ if, and only if, $F(x) = \mathcal{K}_\Psi(F)(x)$, where

$$\mathcal{K}_\Psi(F)(x) := \int_X K_\Psi(x, y) F(y) d\mu(y). \quad (12)$$

In addition, \mathcal{K}_Ψ is the orthogonal projection from $L^2(X, \mu)$ onto $\text{Ran } (C_\Psi)$.

In the following, we will prove that $L^2(X, \mu)$ is not a RKHS if (X, μ) is an-atomic. This statement needs some elaboration to be precise. By definition, $L^2(X, \mu)$ is a space of equivalence classes of functions and does therefore not allow for pointwise evaluation. However, if we take an orthonormal basis $\{\phi_i\}_{i \in \mathcal{I}}$ for $L^2(X, \mu)$ and fix one particular representative for every basis element, then every vector $f \in L^2(X, \mu)$ has a pointwise interpretation via its basis expansion

$$f(x) = \sum_{i \in \mathcal{I}} \langle f, \phi_i \rangle \phi_i(x). \quad (13)$$

If $L^2(X, \mu)$ were a RKHS, then (13) is well defined by Theorem 3.

Proposition 12 *Let (X, μ) be a non-atomic measure space, then $L^2(X, \mu)$ is not a reproducing kernel Hilbert space.*

Proof: Let $A \subset X$ with positive measure. We may assume without loss of generality that $\mu(A) = 1$. Let $\{A_m\}_{m=1}^n$ be a partition of A satisfying $\mu(A_m) = 1/n$, for all $m = 1, \dots, n$. Such a partition exists by Theorem 1. Define $B^n \subset A$ by

$$B^n := \{x \in A : \chi_{A_m}(x) = 1, \text{ for some } m \in \{1, \dots, n\}\}.$$

Clearly, $\mu(B^n) = \mu(A)$ for every $n \in \mathbb{N}$. Let us assume that $L^2(X, \mu)$ is a RKHS, then, for all $x \in B^n$ and some $m \in \{1, \dots, n\}$, one has

$$|\chi_{A_m}(x)|^2 = 1 = |\langle \chi_{A_m}, k_x \rangle|^2 \leq \|k_x\|^2 / n.$$

In particular, $\|k_x\|^2 \geq n$ for all $x \in B^n$. Setting $B := \bigcap_{n \in \mathbb{N}} B^n$ one gets that $\mu(B) = \mu(A)$. Consequently, $\|k_x\|^2 = K(x, x) = \infty$ for almost every $x \in A$, a contradiction to (5). \square

Corollary 13 *Let (X, μ) be an-atomic, then $L^2(X, \mu)$ is not a reproducing kernel Hilbert space.*

Proof: Follows directly from Lemma 2 (ii) and 12. \square

Corollary 14 *Let (X, μ) be an-atomic. There is no orthonormal basis $\{\phi_i\}_{i \in \mathcal{I}} \subset L^2(X, \mu)$, such that*

$$\sum_{i \in \mathcal{I}} |\phi_i(x)|^2 < \infty, \quad \forall x \in X.$$

In particular, for every orthonormal basis $\{\phi_i\}_{i \in \mathcal{I}} \subset L^2(X, \mu)$, there exists a set A of positive measure, such that

$$\sum_{i \in \mathcal{I}} |\phi_i(x)|^2 = \infty, \quad \forall x \in A.$$

Lemma 15 *Let (X, μ) be an-atomic and $\mathcal{H}_K \subset L^2(X, \mu)$ a RKHS, then $\dim(\mathcal{H}_K^\perp) = \infty$.*

Proof: Let us assume that $\dim(\mathcal{H}_K^\perp) = N < \infty$, then any orthonormal basis $\{\phi_i\}_{i \in \mathcal{I}}$ of \mathcal{H}_K can be complemented to a orthonormal basis of $L^2(X, \mu)$ using N vectors $\{u_n\}_{n=1}^N$. In particular, we have by Theorem 3 that

$$\sum_{i \in \mathcal{I}} |\phi_i(x)|^2 + \sum_{n=1}^N |u_n(x)|^2 < \infty, \quad \forall x \in X,$$

which contradicts Corollary 14. \square

With the results of Proposition 12 and 15 we are now ready to give a new proof of Theorem 9.

Proof of Theorem 9: Follows directly from Proposition 11 and 12 and Lemma 15. \square

Remark 16 *The results of this section lead to interesting consequences in the context of quantum mechanics. Let us assume that $\Psi : X \rightarrow \mathcal{H}$ is a system of coherent states, see [3]. The probabilistic interpretation of quantum mechanics then states that the probability distribution of finding a system f in the state $\Psi(x)$ is given by $|\langle f, \Psi(x) \rangle|^2$. Hence, in the light of Theorem 9, it follows that there is a infinite dimensional subspace of probability distributions that does not correspond to any physically feasible system.*

4.2 Strictly continuous mappings

Definition 4 *We call a mapping $\Psi : X \rightarrow \mathcal{H}$ strictly continuous if (X, μ) is non-atomic and there is no set A with $\mu(A) > 0$ such that $C_\Psi f$ is constant on A , for all $f \in \mathcal{H}$.*

Square-integrable group representations [24] like the short-time Fourier system or the continuous wavelet system, see [23], are just one class out of a large reservoir of strictly continuous mappings. In the rest of this section we show that continuous frames can be decomposed into a discrete and a strictly continuous system. To this end, we will need two auxiliary lemmata.

Lemma 17 ([35], Theorem 3.8.1) *Let $A \subset X$ be an atom. Every measurable function $F : X \rightarrow \mathbb{C}$ is constant almost everywhere on A .*

Lemma 18 *Let Ψ be Bessel and $A \subset X$ such that $\mu(A) > 0$ and $\langle f, \Psi(\cdot) \rangle$ is constant on A for every $f \in \mathcal{H}$, then there exists a unique $\psi \in \mathcal{H}$ such that*

$$\|C_\Psi f\|_2^2 = \|C_\Psi f|_{X \setminus A}\|_2^2 + |\langle f, \psi \rangle|^2, \quad \forall f \in \mathcal{H}.$$

In particular, ψ is weakly given by

$$\langle f, \psi \rangle := \mu(A)^{-1/2} \int_A \langle f, \Psi(x) \rangle d\mu(x), \quad \forall f \in \mathcal{H}. \quad (14)$$

Proof: First, observe that ψ defined by (14) is unique for every $n \in \mathbb{N}$ by Riesz representation theorem

$$|\langle f, \psi \rangle| \leq \frac{1}{\sqrt{\mu(A)}} \int_A |\langle f, \Psi(x) \rangle| d\mu(x) \leq \left(\int_A |\langle f, \Psi(x) \rangle|^2 d\mu(x) \right)^{\frac{1}{2}} \leq M \|f\|,$$

where M is the upper frame bound of Ψ . Moreover,

$$\begin{aligned} \int_X |\langle f, \Psi(x) \rangle|^2 d\mu(x) &= \int_{X \setminus A} |\langle f, \Psi(x) \rangle|^2 d\mu(x) + \int_A |\langle f, \Psi(x) \rangle|^2 d\mu(x) \\ &= \int_{X \setminus A} |\langle f, \Psi(x) \rangle|^2 d\mu(x) + |\langle f, \psi \rangle|^2 \end{aligned}$$

where we have used that $\langle f, \Psi(\cdot) \rangle$ is almost everywhere constant on A and (14). \square

Theorem 19 *Every frame Ψ can be written as $\Psi = \Psi_d \cup \Psi_c$, where Ψ_d is a discrete Bessel system and Ψ_c is a strictly continuous Bessel mapping.*

Proof: By Lemma 2 (i), any measure μ can be written as $\mu = \mu_a + \mu_c$, where μ_a is atomic and μ_c is non-atomic. By Lemma 17 and 18 we deduce that Ψ defined on (X, μ_a) can be identified with a discrete Bessel system Ψ_d^a . Let $X_d \subset X$ be the disjoint union of all sets of positive measure with respect to μ_c on which $C_\Psi f$ is constant for all $f \in \mathcal{H}$ and $\{\psi_i\}_{i \in \mathcal{I}}$ the corresponding collection of vectors. By definition $\Psi_c := \Psi|_{X \setminus X_d}$ is strictly continuous. It therefore remains to show that \mathcal{I} is countable. This, however, is a direct consequence from the fact that σ -finite measure spaces can only be partitioned into countably many sets of positive measure. Hence setting $\Psi_d := \Psi_d^a \cup \{\psi_i\}_{i \in \mathcal{I}}$ yield the result. \square

In an attempt to generalize the concept of Riesz bases, continuous Riesz bases [9] and Riesz-type mappings [22] have been introduced. It turns out that these notions are equivalent and characterized by as frames with redundancy zero [9, Proposition 2.5 & Theorem 2.6].

Corollary 20 *Every continuous Riesz basis (Riesz-type mapping) can be written as a discrete Riesz basis.*

Proof: Let Ψ be a continuous Riesz basis, then $R(\Psi) = 0$. By Theorem 9, (X, μ) is atomic. Consequently, Ψ corresponds to a discrete Riesz basis by Lemma 17 and 18. \square

With the results of this section in mind, we suggest to use the term continuous frame only in the case of a strictly continuous frame, and semi-continuous frame if there is both a strictly continuous and a discrete part. Moreover, the notion of continuous Riesz basis/ Riesz type mapping should be discarded as there are no such systems on an-atomic measure spaces and continuous Riesz bases on atomic spaces reduce to discrete Riesz bases.

4.3 Upper semi-frames

In this section we want to illustrate that upper semi-frames behave essentially different from (lower semi-)frames in respect of the problems of Section 4.1. In particular, the closure of the range of the analysis operator is not necessarily a reproducing kernel Hilbert space and there exist upper semi-frames on non-atomic measure spaces with redundancy zero (compare to Proposition 11 and Theorem 9). Throughout this section we will assume that any upper semi-frame violates the lower frame inequality.

Example 1 In [4, 7] the following upper semi-frame has been studied. Set $\mathcal{H}_n := L^2(\mathbb{R}^+, r^{n-1}dr)$, where $n \in \mathbb{N}$, and $(X, \mu) = (\mathbb{R}, dx)$. We use the following convention to denote the Fourier transform

$$\widehat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \omega} dx.$$

Let $\psi \in \mathcal{H}_n$ and define the affine coherent state by

$$\Psi(x)(r) := e^{-2\pi i x r} \psi(r), \quad r \in \mathbb{R}^+, \quad x \in \mathbb{R}.$$

The mapping Ψ forms an upper semi-frame if $\text{ess sup}_{r \in \mathbb{R}^+} \mathfrak{s}(r) < \infty$, where $\mathfrak{s}(r) := r^{n-1} |\psi(r)|^2$, and $|\psi(r)| \neq 0$, for a.e. $r \in \mathbb{R}^+$. The frame operator is then given by a multiplication operator on \mathcal{H}_n , that is,

$$(S_{\Psi} f)(r) = \mathfrak{s}(r) f(r).$$

It is thus easy to see that Ψ cannot form a frame since $\text{ess inf}_{r \in \mathbb{R}^+} \mathfrak{s}(r) = 0$ for every $\psi \in \mathcal{H}_n$. In [7, Section 5.2] it is shown that $\text{Ker } D_{\Psi} = \mathcal{F}_+$, where

$$\mathcal{F}_+ := \{f \in L^2(\mathbb{R}) : \widehat{f}(\omega) = 0, \text{ for a.e. } \omega \geq 0\}.$$

Clearly, $\overline{\text{Ran } C_\Psi} = (\text{Ker } D_\Psi)^\perp = \mathcal{F}_+^\perp = \mathcal{F}_-$, where

$$\mathcal{F}_- := \{f \in L^2(\mathbb{R}) : \widehat{f}(\omega) = 0, \text{ for a.e. } \omega \leq 0\}.$$

Therefore, Ψ has infinite redundancy and a short argument shows that \mathcal{F}_- is not a RKHS:

The dilation operator D_a , defined by $D_a f(x) := a^{-1/2} f(x/a)$, $a \in \mathbb{R}^+$, acts isometrically on \mathcal{F}_- . Take $f \in \mathcal{F}_-$ with $\|f\| = 1$ and $f(0) \neq 0$, then $|D_a f(0)| = |a^{-1/2} f(0)| \rightarrow \infty$, as $a \rightarrow 0$. Consequently, point evaluation cannot be continuous and \mathcal{F}_- is not a RKHS.

The mapping Ψ possesses several other interesting properties, see [7]. For instance, it forms a total Bessel system with no dual. Or, in other words, there is no mapping Φ such that (Ψ, Φ) generates a reproducing pair.

Next, we will show the existence of upper semi-frames with $\text{Ran } C_\Psi$ dense in $L^2(X, \mu)$ if there exists an orthonormal basis of $L^2(X, \mu)$ which is pointwise bounded. In particular, there exist upper semi-frames on non-atomic measure spaces with redundancy zero.

Proposition 21 *Let (X, μ) be a measure space, such that there exists an orthonormal basis $\{\psi_n\}_{n \in \mathbb{N}}$ of $L^2(X, \mu)$ satisfying*

$$\sup_{n \in \mathbb{N}} \sup_{x \in X} |\psi_n(x)| = C < \infty, \quad (15)$$

then there exists an upper semi-frame Ψ for \mathcal{H} such that $\overline{\text{Ran } C_\Psi} = L^2(X, \mu)$. In particular, $R(\Psi) = 0$.

Proof: Take an arbitrary orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of \mathcal{H} , and define

$$\Psi(x) := \sum_{n \in \mathbb{N}} n^{-1} e_n \psi_n(x),$$

with the sum converging absolutely in every point. Then, Ψ is an upper semi-frame with the desired properties. To see this, we first observe that $\Psi : X \rightarrow \mathcal{H}$ is well-defined as, for $x \in X$ fixed,

$$\begin{aligned} |\langle f, \Psi(x) \rangle| &\leq \sum_{n \in \mathbb{N}} |\langle f, e_n \rangle n^{-1} \psi_n(x)| \leq \|f\| \left(\sum_{n \in \mathbb{N}} n^{-2} |\psi_n(x)|^2 \right)^{1/2} \\ &\leq C \|f\| \left(\sum_{n \in \mathbb{N}} n^{-2} \right)^{1/2} = \frac{\pi}{\sqrt{6}} C \|f\|, \end{aligned}$$

where we used (15) and Cauchy-Schwarz inequality. Moreover,

$$\int_X |\langle f, \Psi(x) \rangle|^2 d\mu(x) \leq \int_X \|f\|^2 \sum_{n \in \mathbb{N}} n^{-2} |\psi_n(x)|^2 d\mu(x)$$

$$= \|f\|^2 \sum_{n \in \mathbb{N}} n^{-2} \int_X |\psi_n(x)|^2 d\mu(x) = \|f\|^2 \sum_{n \in \mathbb{N}} n^{-2} = \frac{\pi^2}{6} \|f\|^2.$$

Since $\{\psi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2(X, \mu)$ it follows that Ψ is total in \mathcal{H} as, for $f \neq 0$,

$$\begin{aligned} \int_X |\langle f, \Psi(x) \rangle|^2 d\mu(x) &= \int_X \sum_{n, k \in \mathbb{N}} \langle f, e_n \rangle \langle e_k, f \rangle (nk)^{-1} \psi_n(x) \overline{\psi_k(x)} d\mu(x) \\ &= \sum_{n, k \in \mathbb{N}} \langle f, e_n \rangle \langle e_k, f \rangle (nk)^{-1} \delta_{n, k} = \sum_{n \in \mathbb{N}} |\langle f, e_n \rangle|^2 n^{-2} > 0. \end{aligned}$$

Finally, the range of the analysis operator of the system $\{n^{-1}e_n\}_{n \in \mathbb{N}}$ is dense in $l^2(\mathbb{N})$, which implies that $\text{Ran } C_\Psi$ is dense in $L^2(X, \mu)$. \square

Example 2 Let $(X, \mu) = (\mathbb{T}, dx)$ be the torus with Lebesgue measure, and $\psi_n(x) = e^{2\pi i x n}$, $n \in \mathbb{Z}$. Then, $\{\psi_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis and

$$\sup_{n \in \mathbb{Z}} \sup_{x \in \mathbb{T}} |\psi_n(x)| = 1.$$

Hence, there exists an upper semi-frame Ψ with the closure of $\text{Ran } C_\Psi$ being $L^2(\mathbb{T}, dx)$.

4.4 Correction of the proof of a result on the existence of duals for lower semi-frames

In this section we corrected version of the proof of [4, Proposition 2.6] which states that for every lower semi-frame Ψ there exists a dual mapping Φ such that $S_{\Psi, \Phi} = I$ on $\text{Dom } C_\Psi$. While the result itself is correct, the construction of the dual system Φ in [4] is in general not well-defined. In particular, Φ is defined as

$$\Phi(x) := \sum_{n \in \mathbb{N}} \phi_n(x) V \phi_n = V \left(\sum_{n \in \mathbb{N}} \phi_n(x) \phi_n \right),$$

where $V : L^2(X, \mu) \rightarrow \mathcal{H}$ is a bounded operator depending on Ψ and $\{\phi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis for $L^2(X, \mu)$. However, if (X, μ) is an-atomic, then there exists a set of positive measure A such that $\sum_{n \in \mathbb{N}} |\phi_n(x)|^2 = \infty$, for all $x \in A$, by Corollary 14. Thus, Ψ is not well-defined on a set of positive measure.

Proposition 22 ([4], Proposition 2.6) *Let Ψ be a lower semi-frame, there exists an upper semi-frame Φ such that*

$$f = \int_X \langle f, \Psi(x) \rangle \Phi(x) d\mu(x), \quad \forall f \in \text{Dom } C_\Psi.$$

Proof: Let Ψ be a lower semi-frame, then $\text{Ran } C_\Psi$ is a RKHS in $L^2(X, \mu)$ by Proposition 11. Moreover, let P denote the orthogonal projection from $L^2(X, \mu)$ onto $\text{Ran } C_\Psi$, and $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for \mathcal{H} . Define the linear operator $V : L^2(X, \mu) \rightarrow \mathcal{H}$ by $V := C_\Psi^{-1}$ on $\text{Ran } C_\Psi$ and $V := 0$ on $(\text{Ran } C_\Psi)^\perp$. Then V is bounded and for all $f \in \text{Dom } C_\Psi$, $g \in \mathcal{H}$, it holds

$$\begin{aligned} \langle f, g \rangle &= \langle VC_\Psi f, g \rangle = \langle C_\Psi f, V^*g \rangle_2 = \langle C_\Psi f, V^*(\sum_{n \in \mathbb{N}} \langle g, e_n \rangle e_n) \rangle_2 \\ &= \langle C_\Psi f, \sum_{n \in \mathbb{N}} \langle g, e_n \rangle V^*e_n \rangle_2 = \langle C_\Psi f, \sum_{n \in \mathbb{N}} \langle g, e_n \rangle PV^*e_n \rangle_2 = \langle C_\Psi f, C_\Phi g \rangle_2, \end{aligned}$$

where $\Phi(x) := \sum_{n \in \mathbb{N}} \overline{(PV^*e_n)}(x)e_n$. It remains to show that $\Phi(x)$ is well-defined for every $x \in X$. Since $\{e_n\}_{n \in \mathbb{N}}$ is an orthonormal basis, one has that $\Phi(x)$ is well defined if, and only if,

$$\sum_{n \in \mathbb{N}} |(PV^*e_n)(x)|^2 < \infty, \quad \forall x \in X.$$

By Proposition 6, it is sufficient to show that $\Theta := \{PV^*e_n\}_{n \in \mathbb{N}}$ is a Bessel sequence on $\text{Ran } C_\Psi$. Let $F \in \text{Ran } C_\Psi$, then

$$\sum_{n \in \mathbb{N}} |\langle F, \Theta_n \rangle_2|^2 = \sum_{n \in \mathbb{N}} |\langle VPF, e_n \rangle_2|^2 = \|VF\|^2 \leq C\|F\|_2^2,$$

as $PF = F$ and V is bounded. It hence remains to show that Φ is Bessel. Let $f \in \mathcal{H}$, then

$$\begin{aligned} \int_X |\langle f, \Phi(x) \rangle|^2 d\mu(x) &= \int_X \left| \sum_{n \in \mathbb{N}} \langle f, e_n \rangle \Theta_n(x) \right|^2 d\mu(x) \\ &= \|D_\Theta \{\langle f, e_n \rangle\}_{n \in \mathbb{N}}\|_2^2 \leq C \sum_{n \in \mathbb{N}} |\langle f, e_n \rangle|^2 = C\|f\|^2, \end{aligned}$$

as Θ is Bessel. □

Remark 23 *There is no analogue result of Proposition 22 if Ψ is an upper semi-frame. In [7] it is shown that the affine coherent state system presented in Section 4.3 is a complete Bessel mapping with no dual.*

5 Reproducing pairs and RKHSs

The absence of frame bounds causes problems in the analysis of the ranges of C_Ψ and C_Φ of a reproducing pair (Ψ, Φ) . On the one hand, without the upper frame bound it is no longer guaranteed that $\text{Ran } C_\Psi$ is a subspace of $L^2(X, \mu)$. The lower frame inequality, on the other hand, ensured that

$\text{Ran } C_\Psi$ is a RKHS. A construction of two mutually dual Hilbert spaces intrinsically generated by the pair (Ψ, Φ) is presented in [7]. Let us first recall some of these results before we explain how reproducing kernel Hilbert spaces come into play.

Let $\mathcal{V}_\Phi(X, \mu)$ be the space of all measurable functions $F : X \rightarrow \mathbb{C}$ for which there exists $M > 0$ such that

$$\left| \int_X F(x) \langle \Phi(x), g \rangle d\mu(x) \right| \leq M \|g\|, \quad \forall g \in \mathcal{H}.$$

Note that in general neither $\mathcal{V}_\Phi(X, \mu) \subset L^2(X, \mu)$ nor $L^2(X, \mu) \subset \mathcal{V}_\Phi(X, \mu)$. The linear map $T_\Phi : \mathcal{V}_\Phi(X, \mu) \rightarrow \mathcal{H}$ given weakly by

$$\langle T_\Phi F, g \rangle = \int_X F(x) \langle \Phi(x), g \rangle d\mu(x), \quad g \in \mathcal{H}, \quad (16)$$

is thus well defined by Riesz representation theorem. The operator T_Φ can be seen as the natural extension of the synthesis operator D_Φ (defined on $\text{Dom } D_\Phi \subseteq L^2(X, \mu)$) to $\mathcal{V}_\Phi(X, \mu)$.

Let (Ψ, Φ) be a reproducing pair, according to [7] it then holds

$$\mathcal{V}_\Phi(X, \mu) = \text{Ran } C_\Psi \oplus \text{Ker } T_\Phi. \quad (17)$$

This observation, together with the fact that T_Φ is general not injective, motivates to define the redundancy for arbitrary complete mappings via

$$R(\Phi) := \dim(\text{Ker } T_\Phi). \quad (18)$$

We expect that similar results on $R(\Phi)$ as in Section 4.1 hold.

Conjecture 1 *If $R(\Phi) < \infty$, then (X, μ) is atomic.*

The main difficulty is that there is no characterization of $\mathcal{V}_\Phi(X, \mu)$ which would allow to treat the problem in a similar manner than in Section 4.1 using (17). It is in particular not even clear if $\mathcal{V}_\Phi(X, \mu)$ is normable.

Let us introduce the following vector space

$$V_\Phi(X, \mu) = \mathcal{V}_\Phi(X, \mu) / \text{Ker } T_\Phi,$$

equipped with the inner product

$$\langle F, G \rangle_\Phi := \langle T_\Phi F, T_\Phi G \rangle, \quad \text{where } F, G \in V_\Phi(X, \mu).$$

This is indeed an inner product as $\langle F, F \rangle_\Phi = 0$ if, and only if, $F \in \text{Ker } T_\Phi$. Hence, $V_\Phi(X, \mu)$ forms a pre-Hilbert space and $T_\Phi : V_\Phi(X, \mu) \rightarrow \mathcal{H}$ is an isometry. By (16) $\langle \cdot, \cdot \rangle_\Phi$ can be written explicitly as

$$\langle F, G \rangle_\Phi = \int_X \int_X F(x) \langle \Phi(x), \Phi(y) \rangle \overline{G(y)} d\mu(x) d\mu(y). \quad (19)$$

With the basic definitions at hand, we are now able to give an interpretation of [7, Theorem 4.1] in terms of RKHSs. In particular, this result answers the question if, given a mapping Φ , there exist another mapping Ψ such that (Ψ, Φ) forms a reproducing pair.

Theorem 24 ([7], Theorem 4.1) *Let $\Phi : X \rightarrow \mathcal{H}$ be a weakly measurable mapping and $\{e_i\}_{i \in \mathcal{I}}$ an orthonormal basis of \mathcal{H} . There exists another family Ψ , such that (Ψ, Φ) is a reproducing pair if, and only if,*

$$(i) \text{ Ran } T_\Phi = \mathcal{H},$$

$$(ii) \text{ there exists } \{\mathcal{E}_i\}_{i \in \mathcal{I}} \subset \mathcal{V}_\Phi(X, \mu) \text{ satisfying } T_\Phi \mathcal{E}_i = e_i, \forall i \in \mathcal{I}, \text{ and}$$

$$\sum_{i \in \mathcal{I}} |\mathcal{E}_i(x)|^2 < \infty, \forall x \in X. \quad (20)$$

A reproducing partner Ψ is then given by

$$\Psi(x) := \sum_{i \in \mathcal{I}} \overline{\mathcal{E}_i(x)} e_i. \quad (21)$$

Theorem 24 is a powerful tool for the study of complete systems. It has for example been used to construct a reproducing partner for the Gabor system of integer time-frequency shifts of the Gaussian window [37].

Let us briefly discuss the conditions (i) and (ii). For a complete system one can show that (under very mild conditions [4, Lemma 2.2]) $\overline{\text{Ran } D_\Phi} = \mathcal{H}$ holds. It might therefore seem that (i) is mainly a formality since T_Φ extends D_Φ to its domain $\mathcal{V}_\Phi(X, \mu)$. The upper semi-frame from Section 4.3 however does not satisfy (i), see [7, Section 6.2.3]. In addition, there are intuitive interpretations of (i) and (ii) in different contexts.

Coefficient map interpretation: Property (i) ensures the existence of a linear coefficient map $A : \mathcal{H} \rightarrow \mathcal{V}_\Phi(X, \mu)$ satisfying $f = T_\Phi A(f)$ for every $f \in \mathcal{H}$. Property (ii) then guarantees that $A(f)$ can be calculated taking inner products of f with a second mapping $\Psi : X \rightarrow \mathcal{H}$.

RKHS interpretation: Let us assume that (i) and (ii) are satisfied. The family $\{\mathcal{E}_i\}_{i \in \mathcal{I}}$ forms an orthonormal system with respect to the inner product $\langle \cdot, \cdot \rangle_\Phi$, since by (ii) it holds

$$\langle \mathcal{E}_i, \mathcal{E}_k \rangle_\Phi = \langle T_\Phi \mathcal{E}_i, T_\Phi \mathcal{E}_k \rangle = \langle e_i, e_k \rangle = \delta_{i,k}.$$

Hence, $\{\mathcal{E}_i\}_{i \in \mathcal{I}}$ forms an orthonormal basis for

$$\mathcal{H}_K^\Phi := \overline{\text{span}\{\mathcal{E}_i : i \in \mathcal{I}\}}^{\|\cdot\|_\Phi}.$$

Theorem 3 together with (20) ensure that \mathcal{H}_K^Φ is a RKHS. Moreover, the definition of the reproducing partner Ψ in (21) yields that

$$\mathcal{H}_K^\Phi \simeq V_\Phi(X, \mu) \simeq (\text{Ran } C_\Psi, \|\cdot\|_\Phi). \quad (22)$$

To put it another way, (i) and (ii) guarantee that there exists a RKHS $\mathcal{H}_K^\Phi \subset \mathcal{V}_\Phi(X, \mu)$ which reproduces \mathcal{H} in the sense that $T_\Phi(\mathcal{H}_K^\Phi) = \mathcal{H}$.

Let us assume that (Ψ, Φ) is a reproducing pair. There is a natural way to generate frames on \mathcal{H} and \mathcal{H}_K^Φ using the analysis and synthesis operators.

Proposition 25 *Let (Ψ, Φ) be a reproducing pair for \mathcal{H} , $\{g_i\}_{i \in \mathcal{I}}$ a frame for \mathcal{H} and $\{G_i\}_{i \in \mathcal{I}}$ a frame for \mathcal{H}_K^Φ . Define $H_i(x) := \langle g_i, \Psi(x) \rangle$ and $h_i := T_\Phi G_i$, then $\{H_i\}_{i \in \mathcal{I}}$ is a frame for \mathcal{H}_K^Φ and $\{h_i\}_{i \in \mathcal{I}}$ is a frame for \mathcal{H} .*

Proof: Let $F \in \mathcal{H}_K^\Phi$, then

$$\begin{aligned} \sum_{i \in \mathcal{I}} |\langle F, H_i \rangle_\Phi|^2 &= \sum_{i \in \mathcal{I}} |\langle T_\Phi F, T_\Phi H_i \rangle|^2 = \sum_{i \in \mathcal{I}} |\langle T_\Phi F, S_{\Psi, \Phi} g_i \rangle|^2 \\ &= \sum_{i \in \mathcal{I}} |\langle (S_{\Psi, \Phi})^* T_\Phi F, g_i \rangle|^2 \leq M \|(S_{\Psi, \Phi})^* T_\Phi F\|^2 \\ &\leq M \|S_{\Psi, \Phi}\|^2 \|T_\Phi F\|^2 = \widetilde{M} \|F\|_\Phi^2. \end{aligned}$$

The lower bound follows from the same argument as $(S_{\Psi, \Phi})^*$ is boundedly invertible. Hence, $\{H_i\}_{i \in \mathcal{I}}$ is a frame for \mathcal{H}_K^Φ .

Let $f \in \mathcal{H}$, then

$$\|f\| = \|T_\Phi C_\Psi S_{\Psi, \Phi}^{-1} f\| = \|C_\Psi S_{\Psi, \Phi}^{-1} f\|_\Phi,$$

together with

$$\sum_{i \in \mathcal{I}} |\langle f, h_i \rangle|^2 = \sum_{i \in \mathcal{I}} |\langle T_\Phi C_\Psi S_{\Psi, \Phi}^{-1} f, T_\Phi G_i \rangle|^2 = \sum_{i \in \mathcal{I}} |\langle C_\Psi S_{\Psi, \Phi}^{-1} f, G_i \rangle_\Phi|^2,$$

yields that $\{h_i\}_{i \in \mathcal{I}}$ is a frame for \mathcal{H} . \square

The rest of this section is concerned with the explicit calculation of the reproducing kernel for \mathcal{H}_K^Φ . Let (Ψ, Φ) be a reproducing pair, then there exists a similar characterization of the range of the analysis operators as in (12). Let $F \in \mathcal{V}_\Phi(X, \mu)$ and define $R_{\Psi, \Phi}(x, y) := \langle S_{\Psi, \Phi}^{-1} \Phi(y), \Psi(x) \rangle$ and its associated integral operator

$$\mathcal{R}_{\Psi, \Phi}(F)(x) := \int_X F(y) R_{\Psi, \Phi}(x, y) d\mu(y).$$

By [36, Proposition 2] it follows that $\mathcal{R}_{\Psi, \Phi}(F)(x) = F(x)$ if, and only if, there exists $f \in \mathcal{H}$ such that $F(x) = \langle f, \Psi(x) \rangle$, for all $x \in X$. However,

$R_{\Psi,\Phi}$ is not the reproducing kernel for \mathcal{H}_K^Φ since the reproducing formula is based on the inner product of $L^2(X, \mu)$ and not on $\langle \cdot, \cdot \rangle_\Phi$.

By (22), the reproducing kernel is given by a function $k_x \in \text{Ran } C_\Psi$ such that $F(x) = \langle F, k_x \rangle_\Phi$. Let $F \in \text{Ran } C_\Psi$, applying (19) and the identity $f = T_\Phi C_\Psi S_{\Psi,\Phi}^{-1} f$ yields

$$\begin{aligned} F(x) &= \mathcal{R}_{\Psi,\Phi}(F)(x) = \int_X F(y) \langle \Phi(y), (S_{\Psi,\Phi}^{-1})^* \Psi(x) \rangle d\mu(y) \\ &= \int_X \int_X F(y) \langle \Phi(y), \Phi(z) \rangle \langle \Psi(z), S_{\Psi,\Phi}^{-1} (S_{\Psi,\Phi}^{-1})^* \Psi(x) \rangle d\mu(z) d\mu(y) \\ &= \left\langle F, \langle (S_{\Psi,\Phi}^{-1})^* \Psi(x), (S_{\Psi,\Phi}^{-1})^* \Psi(\cdot) \rangle \right\rangle_\Phi \end{aligned}$$

Hence, using $(S_{\Psi,\Phi}^{-1})^* = S_{\Phi,\Psi}^{-1}$, we finally obtain

$$K_{\Psi,\Phi}(x, y) = k_x(y) = \langle S_{\Phi,\Psi}^{-1} \Psi(x), S_{\Phi,\Psi}^{-1} \Psi(y) \rangle.$$

6 Conclusion

The results of this paper suggest to change the usage of some notions in frame theory. We have shown that any frame can be decomposed into a discrete and a strictly continuous part. In this light, it is reasonable to use the term continuous (semi-)frame only if it is actually strictly continuous and semi-continuous (resp. discrete) frame otherwise. Moreover, since the underlying measure space of a frame with finite is atomic, all efforts to generalize Riesz bases to general measure spaces are condemned to failure from the beginning.

We have investigated the redundancy of (semi-)frames in detail and showed that, in this regard, upper semi-frames may behave essentially different from systems satisfying the lower frame bound. It is an open question to us whether a similar result like Theorem 9 can be proven for the redundancy of a reproducing pair defined in (18).

Another interesting topic for future research is to find and study alternative notions of redundancy for continuous frames. A promising approach that may be adapted can be found in [12]. Studying the dependence on the measure space should thereby remain a key objective.

To sum up, we hope that we could emphasize the fundamental importance of RKHSs for analysis/synthesis processes like frames or reproducing pairs.

Appendix

Proof of Lemma 2: Ad (i): See [18].

Ad (ii): Let (X, μ) be an-atomic. Let us assume on the contrary that for every measurable set $A \subset X$ with $\mu(A) > 0$ there exists an atom $B \subset A$ and let $\{A_n\}_{n \in \mathcal{I}} \subset X$ be a countable partition of X by sets of finite measure. We will show that each A_n can be partitioned into atoms and null sets, a contradiction. Assume without loss of generality that $\mu(A_1) > 0$. By assumption, there exists an atom $B_1 \subset A_1$. If $\mu(B_1) = \mu(A_1)$, then A_1 is an atom. If $0 < \mu(B_1) < \mu(A_1)$, then $\mu(A_1 \setminus B_1) > 0$. Hence, there exists an atom $B_2 \subset A_1 \setminus B_1$ and the preceding argument can be repeated. If one has $\mu(A_1 \setminus (\bigcup_{k=1}^K B_k)) > 0$ for all iteration steps K , then $\mu_K := \mu(\bigcup_{k=1}^K B_k)$ defines a strictly increasing sequence, bounded by $\mu(A_1)$. Hence, μ_K is convergent to some μ^* and the limit equals $\mu(A_1)$. Indeed, if $\mu^* < \mu(A_1)$ then, by assumption, there exists an atom $B^* \subset A_1 \setminus \bigcup_{k \in \mathbb{N}} B_k$ and

$$\mu\left(\bigcup_{k \in \mathbb{N}} B_k \cup B^*\right) > \mu^*,$$

a contradiction. Consequently, $A_1 = \bigcup_{k \in \mathbb{N}} B_k \cup N$, where $N = A_1 \setminus \bigcup_{k \in \mathbb{N}} B_k$ is of measure zero. In particular, we have constructed a partition of A_1 consisting of atoms and null sets. Repeating this argument for all A_n , $n \in \mathcal{I}$, with $\mu(A_n) > 0$ shows that (X, μ) is atomic, a contradiction. \square

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